

Symmetries of the Burgers Turbulence without Pressure

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We investigate local symmetries of the Burgers turbulence driven by an external random force. By using a path integral formalism, we show that the Jacobian has physics in it; local symmetries and an anomaly. We also study a local invariance of the effective action and show it is related to Kolmogorov's second law of self-similarity.

I. BURGERS EQUATION

In this paper we investigate one-dimensional Burgers equation

$$u_t + uu_x - \nu u_{xx} = f(t, x), \quad (1)$$

which is driven by Gaussian random force $f(t, x)$,

$$\langle f(t, x)f(t', y) \rangle = \kappa(x - y)\delta(t - t'). \quad (2)$$

Function κ defines the spatial correlation of the random forces and ν is the viscosity. The equation has been studied in several papers, see for example Refs. 1-5.

We use the path integral formalism and write equations (1)-(2) as

$$\langle F[\lambda] \rangle = \int D\mu Du F[\lambda] J[u] \exp(-S[u, \mu]). \quad (3)$$

Action S is defined as

$$S[u, \mu] = \frac{1}{2} \int dt dx dy \mu(t, x) \kappa(x - y) \mu(t, y) - i \int dt dx \mu (u_t + uu_x - \nu u_{xx}), \quad (4)$$

and Jacobian is

$$J[u] = \det \left| \frac{\delta f}{\delta u} \right| = \det |\partial_t + u_x + u\partial_x - \nu\partial_{xx}|. \quad (5)$$

We also use the following representation for the determinant,

$$J[u] = \int D\bar{\Psi} D\Psi \exp(-S_A), \quad (6)$$

where action is

$$S_A = - \int dt dx \bar{\Psi} (\partial_t + u_x + u\partial_x - \nu\partial_{xx}) \Psi. \quad (7)$$

Fields $\Psi = \Psi(t, x, u)$ and $\tilde{\Psi} = \tilde{\Psi}(t, x, u)$ are anticommuting functions, Refs. 6-7.

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II. DETERMINANT

In this section we study local symmetries of determinant action (7) and calculate the Jacobian. When the viscosity is set to zero, action (7) is invariant under local time reparametrization, $\alpha = \alpha(t)$,

$$\delta u_{A1} = \alpha u_t + \alpha' u, \quad (8)$$

$$\delta \Psi_{A1} = \alpha \Psi_t, \quad (9)$$

$$\delta \bar{\Psi}_{A1} = \alpha \bar{\Psi}_t. \quad (10)$$

Action (7) is also invariant under local space-time symmetry, $\epsilon = \epsilon(t, x)$, $\nu = 0$,

$$\delta u_{A2} = \epsilon u_x - \epsilon_x u - \epsilon_t, \quad (11)$$

$$\delta \Psi_{A2} = \epsilon \Psi_x + \epsilon_x \Psi, \quad (12)$$

$$\delta \bar{\Psi}_{A2} = \epsilon \bar{\Psi}_x. \quad (13)$$

Variation with respect to u gives, $\nu = 0$,

$$\frac{\delta S_A}{\delta u} = \bar{\Psi}_x \Psi - \frac{\delta(\bar{\Psi} \Psi_t)}{\delta u} + u \frac{\delta(\bar{\Psi}_x \Psi)}{\delta u}. \quad (14)$$

If assuming that fields Ψ and $\bar{\Psi}$ are independent on u , variation (14) is still nonvanishing. Alternatively, if variation (14) is claimed to be zero, field u has a constraint. This shows that action (7) depends on field u and determinant (5) is field u dependent. The problem here is how to calculate the determinant in such a manner that the causality is preserved.

The reason for the ambiguity is that in the current situation operator in $\delta f / \delta u$ is non-self-adjoint. According to the definition $\det |\delta f / \delta u|$ is the product of the eigenvalues $\delta f / \delta u$. A way to calculate the determinant is to consider the following eigenvalue equations,

$$\begin{cases} (\partial_t + u_x + u \partial_x - \nu \partial_{xx}) A = \lambda A, \\ (-\partial_t - u \partial_x - \nu \partial_{xx}) A = \lambda A, \end{cases} \quad (15)$$

$$\quad (16)$$

where $\lambda = \lambda(t, x, u)$ is an eigenvalue and $A = A(t, x, u)$ is the corresponding eigenfunction. This gives a result

$$\lambda = \frac{1}{2} u_x - \nu \frac{\partial_{xx} A}{A} = \frac{1}{2} u_x - \nu G[u], \quad (17)$$

and $G[u]$ is an u -dependent function. When using the result, the full determinant action takes the form

$$S_D = - \int dt dx \bar{\Psi} \left(\frac{1}{2} u_x - \nu G[u] \right) \Psi, \quad (18)$$

or

$$J[u] = \exp \left(\int dt dx \ln \left(\frac{1}{2} u_x - \nu G[u] \right) \right) = \exp(-\bar{S}_D). \quad (19)$$

The effective action is then

$$S_{eff}[u, \mu] = \frac{1}{2} \int dt dx dy \mu(t, x) \kappa(x - y) \mu(t, y) - i \int dt dx \mu (u_t + u u_x - \nu u_{xx}) - \int dt dx \ln \left(\frac{1}{2} u_x - \nu G[u] \right). \quad (20)$$

We have therefore shown that generally Burgers equation has a non-zero Jacobian.

III. ANOMALY

We take a look of action (18). It can be written as

$$S_D = \frac{1}{2} \int dt dx u (\bar{\Psi}_x \Psi + \bar{\Psi} \Psi_x) + \nu \int dt dx G[u] \bar{\Psi} \Psi, \quad (21)$$

$$= \frac{S_A + S_B}{2}, \quad (22)$$

where

$$S_B = - \int dt dx \bar{\Psi} (-\partial_t - u \partial_x - \nu \partial_{xx}) \Psi. \quad (23)$$

The full determinant action (21) is therefore a sum of two non-self-adjoint operators S_A and S_B .

Action (23) is invariant under local time reparametrization, $\alpha = \alpha(t)$,

$$\delta u_{B1} = \alpha u_t + \alpha' u, \quad (24)$$

$$\delta \Psi_{B1} = \alpha \Psi_t, \quad (25)$$

$$\delta \bar{\Psi}_{B1} = \alpha \bar{\Psi}_t, \quad (26)$$

and under local space-time symmetry, $\epsilon = \epsilon(t, x)$, $\nu = 0$,

$$\delta u_{B2} = \epsilon u_x - \epsilon_x u + \epsilon_t, \quad (27)$$

$$\delta \Psi_{B2} = \epsilon \Psi_x, \quad (28)$$

$$\delta \bar{\Psi}_{B2} = \epsilon \bar{\Psi}_x + \epsilon_x \bar{\Psi}. \quad (29)$$

Now we can also rewrite the correct variation for the full determinant, $\nu = 0$,

$$\frac{\delta S_D}{\delta u} = \frac{1}{2} \left[\partial_x (\bar{\Psi} \Psi) + u \frac{\delta \partial_x (\bar{\Psi} \Psi)}{\delta u} \right]. \quad (30)$$

As a summary, if the action of the Burgers equation is claimed to be a number, we have a constraint for field u in Eq. (30).

IV. LOCAL SYMMETRY

In this section we investigate local symmetries of the effective action (20). We consider the following local time reparametrization, $\beta = \beta(t)$, a and b are constants,

$$\tilde{t} = \beta(t)^a, \quad (31)$$

$$\tilde{x} = x \beta'(t)^b, \quad (32)$$

$$\tilde{u} = u \beta'(t)^{a-b}, \quad (33)$$

$$\tilde{\mu} = \mu \beta'(t)^{2b-a}, \quad (34)$$

$$\tilde{\Psi} = \Psi \beta'(t)^{b/2}, \quad (35)$$

$$\tilde{\bar{\Psi}} = \bar{\Psi} \beta'(t)^{b/2}. \quad (36)$$

This translates to the following field variations,

$$\delta u = a\beta u_t + b\beta' x u_x + (a-b)\beta' u, \quad (37)$$

$$\delta \mu = a\beta \mu_t + b\beta' x \mu_x + (2b-a)\beta' \mu, \quad (38)$$

$$\delta \Psi = a\beta \Psi_t + b\beta' x \Psi_x + \frac{b}{2}\beta' \Psi, \quad (39)$$

$$\delta \bar{\Psi} = a\beta \bar{\Psi}_t + b\beta' x \bar{\Psi}_x + \frac{b}{2}\beta' \bar{\Psi}. \quad (40)$$

The transformation is based on symmetries (8)-(13) and (24)-(29) where $\alpha(t) = a\beta(t)$ and $\epsilon(t, x) = b\beta'(t)x$.

We first apply variations (37), (39), (40) to action (18) ($\nu = 0$). This leaves the action invariant, $\delta S_D = 0$. It follows that for consistency reasons also variation of action (19) must be zero when $\nu = 0$,

$$\delta \bar{S}_D = -\frac{1}{2} \int dt dx \frac{\beta'}{u_x} \partial_x (b x u_x + (a-b)u) = 0. \quad (41)$$

The invariance requirement creates a constraint for field u ,

$$b x u_x + (a-b)u = g(t), \quad (42)$$

where $g(t)$ is a time dependent function.

Transformation (37)-(40) for the action S gives

$$\delta S = \frac{3h-1}{2}b \int dt dx dy \beta' \mu(x) \kappa(x-y) \mu(y) - iC \int dt dx \mu + i(h+1)\nu b \int dt dx \beta' \mu u_{xx}, \quad (43)$$

where $h = (b-a)/b$. Here we have used constraint (42) with $g(t) = C/\beta''$ and C is a constant. The pump term is zero when $h = 1/3$. By setting $h = -1$ (or $\nu = 0$) the viscosity term is zero. These situations are explained in more detail in Ref. 8. The second integral is also zero based on the conservation of the center-of-mass motion, Ref. (5).

Constraint (42) defines a solution for field u ,

$$u(t, x) = \tilde{g}(t)x^h - \frac{C}{bh\beta''(t)}, \quad (44)$$

where $\tilde{g}(t)$ is a time dependent function. Variation of u gets the form

$$\delta u(t, x) = a\beta(t)u_t(t, x) + C \frac{\beta'(t)}{\beta''(t)}. \quad (45)$$

Hence we have shown that in the limit of $\nu = 0$, symmetry transformation (31)-(36) is a local symmetry for the effective action (20).

Our interpretation of this symmetry is, as it is explained in Ref. 8, that in the limit of infinite Reynolds numbers, $\nu \rightarrow 0$, all the symmetries of the Navier-Stokes equation are restored in a statistical sense. The condition $\tilde{u}(\tilde{x}) = u(x)$ is also known as a Kolmogorov's second law of self-similarity, Ref. 8.

V. CONCLUSIONS

In this paper we have studied local symmetries of the Burgers equation. By using a path integral formalism and the exact symmetries of the Jacobian we calculated the determinant. We then investigated a local symmetry of the effective action which relates to Kolmogorov's second law of self-similarity. We found an anomaly for the Burgers turbulence which shows as a constraint for the velocity field.

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